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Systems & Control Letters 37 (1999) 87–91

SYSTEMS
& CONTROL
LETTERS

Computing the L_2-induced norm of a compression operator

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Received 22 August 1997; received in revised form 1 February 1999

Abstract

A new computationally viable approach is derived for computing the induced norm of a state space compression operator; that is an integral operator defined on the *finite* length space $L_2[0,h]$. Determining this norm is a crucial component in the control analysis of both delay systems and sampled-data systems. The technique developed may have application to a wider variety of integral operators. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Compression operator; H_∞-norm; delay systems; Sampled-data systems; Integral operators

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1. Introduction

In this paper we provide an explicit and simple procedure for computing the norm of a state-space compression operator. Specifically, we are concerned with determining the induced norm of an operator $\bar{K}$ on the standard Hilbert space $L_2[0,h]$, where $\bar{K}$ is given by

$$(\bar{K}u)(t) = \int_0^t \bar{C}\mathrm{e}^{\bar{A}(t-\tau)}\bar{B}u(\tau)\,\mathrm{d}\tau + \bar{D}u(t) \quad \text{for } 0 \leqslant t \leqslant h. \tag{1}$$

Here $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are appropriately dimensioned real matrices, and h is a positive real number. It is well-known (see for instance [3]) how to compute the $L_2[0,\infty)$ induced norm of an operator of the above form, but at present there is no simple method for accomplishing this for the finite interval space $L_2[0,h]$. The operator $\bar{K}$ is called the compression of the related mapping which acts on $L_2[0,\infty)$.

Computing the norm of a compression operator is an important calculation in a number of control theory problems, most notably in optimal control of time-delay systems (see e.g. [5,7,11]) and sampled-data systems (for instance [1,8,9]). Explicitly computing this norm has been considered in [5,6,11]. These papers provide procedures for finding the norm of $\bar{K}$ by reducing the problem to one of determining the roots of a *transcendental* equation with respect to a real parameter; these procedures also require some technical assumptions such as $\bar{A}$ being Hurwitz. In this note we develop a method to compute the norm of $\bar{K}$ which simply reduces to checking the maximum eigenvalue of an explicit symmetric *matrix*; our approach makes no assumptions about the state-space matrices which define the compression $\bar{K}$.

2. Solution

Throughout we fix the state-space matrices $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$, and the domain length $h > 0$ of the elements in $L_2[0,h]$. In the sequel we use the abbreviation L_2 for $L_2[0,h]$.

With $\bar{K}$ thus defined we aim to answer the following question:

does $\|\bar{K}\|_{L_2\to L_2} < 1$ hold?

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PII: S0167-6911(99)00009-2

Namely, is the operator $\bar{K}$ contractive. Note that for convenience we consider a normalized problem, and that the general problem of answering whether $\|\bar{K}\|_{L_2\to L_2} < \gamma$ is satisfied, for a given $\gamma > 0$, can be accomplished by scaling $\bar{K}$. Thus solving this problem immediately gives rise to a bisection algorithm for computing $\|\bar{K}\|_{L_2\to L_2}$. Observe from (1) that the maximum singular value of $\bar{D}$ must satisfy $\bar{\sigma}(\bar{D}) < 1$ if $\bar{K}$ is contractive. We therefore make the *standing* assumption that the matrix condition $\bar{\sigma}(\bar{D}) < 1$ is satisfied.

Our first step to answering the above question is to use a more convenient representation for $\bar{K}$, one defined on the square summable sequences ℓ_2 instead of L_2. To facilitate this we define the following set of functions in L_2: for a fixed $\theta \in (-\pi, \pi]$ let the sequence of functions $\{\psi_k\}_{k=0}^{\infty}$ be defined by

$$\psi_k(t) = \left(\frac{1}{\sqrt{h}}\right) \mathrm{e}^{\mathrm{j}\omega_k t} \quad \text{for } 0 \leqslant t \leqslant h,$$

where

$$\omega_k := \frac{2\pi v_k + \theta}{h},$$
$$\{v_k\} := \{0, 1, -1, 2, -2 \ldots\}.$$

The latter two sequences defined above will be used frequently in the sequel. We now state a basic fact about the function sequence ψ_k.

Proposition 1. *The set of functions $\{\psi_k\}$ form a complete orthonormal basis for L_2.*

This is straightforward to show, and a proof can be found in [4].

We now define the operator $K : \ell_2 \to \ell_2$ by the inner product relationship

$$\langle e_l, K e_k\rangle := \langle \psi_l, \bar{K}\psi_k\rangle,$$

where $\{e_k\}$ is the standard basis for ℓ_2. Thus we see that K is just $\bar{K}$ represented in the $\{\psi_k\}$ basis. Using this new representation K is motivated by the following result.

Theorem 2. *Suppose* $\mathrm{e}^{\mathrm{j}\theta} \in \mathrm{eig}(\mathrm{e}^{\bar{A}h})$. *Then*

$$K = \begin{bmatrix} C_0 \\ C_1 \\ \vdots \end{bmatrix} (I\mathrm{e}^{\mathrm{j}\theta} - A)^{-1} \begin{bmatrix} B_0 & B_1 & B_2 & \cdots \end{bmatrix} + \begin{bmatrix} G_0 & & & 0 \\ & G_1 & & \\ & & G_2 & \\ 0 & & & \ddots, \end{bmatrix},$$

where

$$\begin{aligned}
A &= \mathrm{e}^{\bar{A}h},\\
B_k &= (I\mathrm{j}\omega_k - \bar{A})^{-1}(I\mathrm{e}^{\mathrm{j}\theta} - A)\bar{B}h^{-1/2},\\
C_k &= \bar{C}(A\mathrm{e}^{-\mathrm{j}\theta} - I)h^{-1/2}(I\mathrm{j}\omega_k - \bar{A})^{-1},\\
G_k &= \bar{C}(I\mathrm{j}\omega_k - \bar{A})^{-1}\bar{B} + \bar{D}.
\end{aligned} \tag{2}$$

This result states that K is the sum of a finite rank operator and a block diagonal one. Note that setting $\widehat{G}(s) := \bar{C}(Is - \bar{A})^{-1}\bar{B} + \bar{D}$, the usual transfer function associated with the state-space matrices, we have $G_k = \widehat{G}(j\omega_k)$. Also realize therefore that θ can always be chosen to satisfy the supposition of the theorem.

Proof. The result follows by a routine state-space calculation of $\langle \psi_l, \bar{K}\psi_k\rangle$ given the definitions of $\bar{K}$ and ψ_k. □

We know that $\|\bar{K}\|_{L_2\to L_2} < 1$ is satisfied if and only if the operator $I - K^*K$ is positive definite,[1] and it is the latter object with which we will work. First we have

$$\begin{aligned}
K^*K &= G^*G + \begin{bmatrix} G^*C & B^* \end{bmatrix} \\
&\quad \times \begin{bmatrix} -C^*C & I\mathrm{e}^{-\mathrm{j}\theta} - A^* \\ I\mathrm{e}^{\mathrm{j}\theta} - A & 0 \end{bmatrix}^{-1} \begin{bmatrix} C^*G \\ B \end{bmatrix} \\
&= \begin{bmatrix} G_0^*G_0 & & 0 \\ & G_1^*G_1 & \\ 0 & & \ddots \end{bmatrix} \\
&\quad + \begin{bmatrix} P_0^* \\ P_1^* \\ \vdots \end{bmatrix} Q^{-1} \begin{bmatrix} P_0 & P_1 & \cdots \end{bmatrix},
\end{aligned} \tag{3}$$

where P is defined by

$$P_k = \begin{bmatrix} C_k^*G_k \\ B_k \end{bmatrix}.$$

We now define a useful condition.

Condition 3. *A positive integer N satisfies this condition if, for each $k > N$, the inequality*

$\bar{\sigma}(G_k) < 1$ *is satisfied.*

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[1] A self-adjoint operator Q is positive definite if there exists an $\alpha > 0$, such that $\langle x, Qx\rangle \geqslant \alpha\langle x, x\rangle$ for all x.

By assumption $\bar{\sigma}(\bar{D}) < 1$ and therefore there always exists an N satisfying the above condition; to see this observe that $\lim_{k\to\infty} G_k = \bar{D}$.

Our next step is to partition the operator K^*K using a fixed number N satisfying the above condition. Let

$$J_0 := \begin{bmatrix} G_0^*G_0 & & 0 \\ & \ddots & \\ 0 & & G_N^*G_N \end{bmatrix},$$

$$J_1 := \begin{bmatrix} G_{N+1}^*G_{N+1} & & 0 \\ & G_{N+2}^*G_{N+2} & \\ 0 & & \ddots \end{bmatrix}$$

and observe that $G^*G = \operatorname{diag}(J_0, J_1)$. The reason for this partitioning of G^*G is that if N satisfies Condition 3 then $I - J_1$ is a positive-definite operator. Further, define the operators F_0 and F_1 satisfying

$$[F_0 \quad F_1] = P,$$

where this partition is chosen so that (3) becomes

$$K^*K = \begin{bmatrix} J_0 & 0 \\ 0 & J_1 \end{bmatrix} + \begin{bmatrix} F_0^* \\ F_1^* \end{bmatrix} Q^{-1}[F_0 \quad F_1].$$

Having established this notation we see that $\bar{K}$ is contractive if and only if

$$0 < I - J^* - F^*Q^{-1}F,$$

where J and F are defined in the obvious way. We can now state the main result of the section and indeed the paper.

Theorem 4. *Suppose* $e^{j\theta} \notin \operatorname{eig}(e^{\bar{A}h})$ *and* N *satisfies Condition* 3. *Then* $\|\bar{K}\|_{L_2\to L_2} < 1$ *holds if and only if*

$$0 < \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} J_0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} F_0^* \\ E_1^* \end{bmatrix} Q^{-1}[F_0 \quad E_1], \tag{4}$$

where E_1 *is any matrix satisfying* $E_1E_1^* = F_1(I - J_1)^{-1}F_1^*$.

This result states that the property of $\bar{K}$ being contractive is equivalent to a matrix condition. Recall from the above discussion that θ and N can always be chosen a priori to satisfy the conditions of the hypothesis.

Proof. Since K is a representation of $\bar{K}$ we have that $\|\bar{K}\|_{L_2\to L_2} < 1$ is satisfied if and only if $0 < I - K^*K$, which is

$$\begin{aligned} 0 &< \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} J_0 & 0 \\ 0 & J_1 \end{bmatrix} - \begin{bmatrix} F_0^* \\ F_1^* \end{bmatrix} Q^{-1} \begin{bmatrix} F_0 & F_1 \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & I - J_1 \end{bmatrix} - \begin{bmatrix} J_0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} F_0^* \\ F_1^* \end{bmatrix} Q^{-1}[F_0 \quad F_1] \end{aligned}$$

from the definitions made so far. Since N satisfies Condition 3 we know $I - J_1 > 0$, and therefore $S := (I - J_1)^{-1/2}$ is well defined. Now pre- and post-multiply the last inequality by

$$\begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix}$$

to get the equivalent condition

$$\begin{aligned} 0 &< \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} J_0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} F_0^* \\ SF_1^* \end{bmatrix} Q^{-1}[F_0 \quad F_1S] \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & SF_1^* \end{bmatrix} \\ &\quad \times \underbrace{\begin{bmatrix} J_0 + F_0^*Q^{-1}F_0 & F_0^*Q^{-1} \\ Q^{-1}F_0 & Q^{-1} \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} I & 0 \\ 0 & F_1S \end{bmatrix}}_{N}. \end{aligned}$$

The last inequality is equivalent to saying that the spectral radius condition $\rho(N^*TN) < 1$ holds. By elementary properties of the spectrum (see, e.g., [2]) this is tantamount to $\rho(TNN^*) < 1$. The operator NN^* is given by

$$\begin{bmatrix} I & 0 \\ 0 & F_1(I - J_1)^{-1}F_1^* \end{bmatrix}.$$

Let E_2 be any matrix satisfying $E_2E_2^* = F_1(I - J_1)^{-1}F_1^*$ and define

$$E = \begin{bmatrix} I & 0 \\ 0 & E_2 \end{bmatrix}.$$

Clearly, $EE^* = NN^*$ and therefore $\rho(N^*TN) < 1$ holds if and only if $\rho(E^*TE) < 1$ holds. Observe that the latter condition is equivalent to that of the theorem. □

We now turn to the computational issues of using the above result.

3. Explicit computation

Theorem 4 provides a matrix test for determining contractiveness of $\bar{K}$, provided one can compute the matrices in the inequality. We have already provided state-space formulae for most of these, but it remains to compute C^*C and $F_1(I-J_1)^{-1}F_1^*$. This is the topic of the current section, and we accomplish this task using ideas from [4].

First we focus on computing $F_1(I-J_1)^{-1}F_1^*$. Observe from its definition that J_1 is a block diagonal operator and therefore so is $(I-J_1)^{-1}$. Also F_1 is a block "row" operator and therefore we see

$$F_1(I-J_1)^{-1}F_1^* = \sum_{k=N+1}^{\infty} P_k(I-G_k^*G_k)^{-1}P_k^*.$$

By routine state-space manipulations each term in this sum is of the form

$$P_k(I-G_k^*G_k)^{-1}P_k^* = C_L(I\mathrm{j}\omega_k - A_L)^{-1}B_L, \tag{5}$$

where C_L, A_L and B_L are matrices given in the appendix. Similarly, we can show that

$$C^*C = \sum_{k=0}^{\infty} C_M(I\mathrm{j}\omega_k - A_M)^{-1}B_M. \tag{6}$$

See the appendix for the matrices C_M, A_M and B_M. Here we assume formally that the inverses in both (5) and (6) are always well defined, and will later show how θ can be chosen a priori to ensure these inverses exist.

To evaluate these series we use the following result.

Proposition 5. *Suppose that* $\mathrm{e}^{\mathrm{j}\theta}$ *is not an eigenvalue of* e^{Wh}, *where* W *is a square matrix. Then*

$$\sum_{k=0}^{\infty}(I\mathrm{j}\omega_k - W)^{-1} = -\mathrm{j}\frac{h}{2}\cot((\mathrm{j}Wh+\theta I)/2).$$

For completeness we provide a proof of this formula, taken from [4].

Proof. Consider the function $W(t) := h(I-\mathrm{e}^{Wh}\mathrm{e}^{-\mathrm{j}\theta})^{-1}\mathrm{e}^{(W-\mathrm{j}\theta/h)t}$ on the interval $[0,h]$: its Fourier series is

$$W(t) \sim \sum_{k=0}^{\infty}(I\mathrm{j}\omega_k - W)^{-1}\mathrm{e}^{\mathrm{j}(2\pi/h)v_k t},$$

where again v_k is the sequence $\{0,1,-1,2,-2,\ldots\}$. Because $W(t)$ is of bounded variation and continuous on the interval, it is well known from classical Fourier analysis that at $t=0$ the series sums to the average of the end points $(W(0)+W(h))/2$, which is just $-j(h/2)\cot((\mathrm{j}Wh+\theta I)/2)$. □

Corollary 6. *Suppose* $\mathrm{e}^{\mathrm{j}\theta}$ *is not an eigenvalue of either* $\mathrm{e}^{A_M h}$ *or* $\mathrm{e}^{A_L h}$. *Then the following formulae hold*:

$$C^*C = -\mathrm{j}C_M\frac{h}{2}\cot((\mathrm{j}A_M h+\theta I)/2)B_M,$$

$$F_1(I-J_1)^{-1}F_1^* = -\mathrm{j}C_L\frac{h}{2}\cot((\mathrm{j}A_L h+\theta I)/2)B_L - \sum_{k=0}^{N} C_L(Ij\omega_k - A_L)^{-1}B_L.$$

Proof. This result follows immediately from the Proposition, since the hypothesis guarantees that the inverses $(I\mathrm{j}\omega_k - A_L)^{-1}$ and $(I\mathrm{j}\omega_k - A_M)^{-1}$ exist for all $k \geqslant 0$. □

4. An algorithm

This section is devoted to collecting the above results into a straightforward algorithm for determining whether $\|\bar{K}\| < 1$ holds. Note that the only specialized operations required in this procedure are the matrix exponential and eigenvalue computation. Most importantly no conditions are imposed on the state-space matrices $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$, other than the necessary condition that $\bar{\sigma}(\bar{D}) < 1$.

The procedure now described appeals to the result in Theorem 4; the steps in this algorithm are used to form the required objects in the final computation. The initial step is to find a parameter θ that satisfies the hypothesis in Corollary 6.

Step 1: select a value for θ: Choose $\theta \in (-\pi,\pi]$ such that $\mathrm{e}^{\mathrm{j}\theta}$ is not a eigenvalue of $\mathrm{e}^{A_L h}$; note this also ensures it is not an eigenvalue of $\mathrm{e}^{A_M h}$.

The next step is to find an N that satisfies Condition 3. We accomplish this easily by invoking a well-known result on computing the L_∞-norm of a transfer function (see for instance [10]): for ω fixed, the singular-value equation $\bar{\sigma}(\hat{G}(\mathrm{j}\omega)) = 1$ holds, if and only if, $\mathrm{j}\omega$ is an eigenvalue of A_G. Here A_G is the Hamiltonian matrix given in Appendix A, and $\hat{G}(s) := \bar{C}(Is-\bar{A})^{-1}\bar{B}+\bar{D}$.

Step 2: choose N: Let $\mu = \max\{|\lambda| : \lambda$ is a purely imaginary eigenvalue of $A_G\}$. Set $N \geqslant 0$ to a value satisfying both $\omega_{N+1} > \mu$ and $\omega_{N+2} > \mu$.

To see that this step yields a satisfactory N observe that for any $|\omega| > \mu$ it must be the case that $\bar{\sigma}(\hat{G}(\mathrm{j}\omega)) < 1$, since $\lim_{\omega\to\infty}\bar{\sigma}(\hat{G}(\mathrm{j}\omega)) = \bar{\sigma}(\bar{D}) < 1$.

Now that the conditions required for the computations have been met, the final step assembles all the data.

Step 3: apply Theorem 4

(i) Compute C^*C and $F_1(I - J_1)^{-1}F_1^*$ using the formulae in Corollary 6;
(ii) Compute Q, F_0, J_0 and E_1 of Theorem 4;
(iii) Determine whether matrix inequality (4) holds.

The above 3-stage algorithm can be easily implemented in software using standard routines. An important feature of this procedure is that it definitively determines whether the operator $\bar{K}$ is contractive, and can thus be used as the foundation of a bisection algorithm to determine the norm of $\bar{K}$ to any desired degree of accuracy.

Acknowledgements

The author thanks Kentaro Hirata for his detailed review of this paper, and pointing out an error in the original state-space formulae.

Appendix. State-space formulae

The state-space matrices required in Eqs. (5) and (6) are provided below. These reduced formulae have been obtained by routine manipulations starting with the state-space for $P_k(I - G_k^*G_k)^{-1}P_k^*$ and $C_k^*C_k$ derived from (2):

$$A_M = \begin{bmatrix} \bar{A} & 0 \\ -\bar{C}^*\bar{C} & -\bar{A}^* \end{bmatrix}, \qquad B_M = \begin{bmatrix} \bar{T} \\ 0 \end{bmatrix},$$

$$C_M = \begin{bmatrix} 0 & \bar{T}^* \end{bmatrix},$$

$$A_L = \begin{bmatrix} \bar{A}_0 & \bar{B}\bar{R}\bar{B}^* & \bar{B}\bar{R}\bar{D}^*\bar{C} & -\bar{B}\bar{R}\bar{B}^* \\ -\bar{C}^*\bar{H}\bar{C} & -\bar{A}_0^* & -\bar{C}^*\bar{D}\bar{R}\bar{D}^*\bar{C} & \bar{C}^*\bar{H}\bar{D}\bar{B}^* \\ 0 & 0 & \bar{A} & 0 \\ 0 & 0 & \bar{C}^*\bar{C} & -\bar{A}^* \end{bmatrix},$$

$$B_L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \bar{T} & 0 \\ 0 & -\mathrm{e}^{-\mathrm{j}\theta}\bar{T}^* \end{bmatrix},$$

$$C_L = \begin{bmatrix} 0 & \bar{T}^* & 0 & 0 \\ -\mathrm{e}^{\mathrm{j}\theta}\bar{T} & 0 & 0 & 0 \end{bmatrix},$$

$$A_G = \begin{bmatrix} \bar{A}_0 & \bar{B}\bar{R}\bar{B}^* \\ -\bar{C}^*\bar{H}\bar{C} & -\bar{A}_0^* \end{bmatrix},$$

where these matrices are further defined in terms of

$$\bar{A}_0 = \bar{A} + \bar{B}\bar{D}^*(I - \bar{D}\bar{D}^*)^{-1}\bar{C},$$

$$\bar{T} = (\mathrm{e}^{\bar{A}h}\mathrm{e}^{-\mathrm{j}\theta} - I)h^{-1/2},$$

$$\bar{R} = (I - \bar{D}^*\bar{D})^{-1},$$

$$\bar{H} = (I - \bar{D}\bar{D}^*)^{-1}.$$

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